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EXTENSION OF A FLOW ON A COMPLETELY REGULAR SPACE

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Abstract. Let X be a completely regular space and let $\beta(X)$ be the Stone-Čech compactification of X. A flow ϕ on X can be extended to a flow Φ on $\beta(X)$.

1. Introduction

Let X be a locally compact Hausdorff space and let X^* be the one-point compactification of X. By defining $\phi^*(*,t) = *$ for all $t \in \mathbb{R}$ a flow ϕ on X is extended to a flow ϕ^* on X^*. Let X be a completely regular space and let $\beta(X)$ be the Stone-Čech compactification of X. It is natural to ask that can a flow ϕ on X be extended to a flow Φ on $\beta(X)$? This paper is a partial answer to this question.

2. Main results

Definition 2.1. Let X be a topological space. A *flow* ϕ on X is a continuous function $\phi : X \times \mathbb{R} \to X$ such that

1. $\phi(x, 0) = x$ for all $x \in X$ and
2. $\phi(\phi(x, s), t) = \phi(x, s + t)$ for all $x \in X$ and all $s, t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, we define a function $\phi^t : X \to X$ by $\phi^t(x) = \phi(x, t)$ for all $x \in X$. Then ϕ^t is a homeomorphism.

Let X be a locally compact Hausdorff space and let ϕ be a flow on X. We define the one-point compactification of ϕ to be the map $\phi^* : X^* \times \mathbb{R} \to X^*$ defined by $\phi^*(x, t) = \phi(x, t)$ if $x \neq *$ and $\phi^*(*, t) = *$ for all $t \in \mathbb{R}$.

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Theorem 2.2. [1] *The one-point compactification ϕ^* of a flow ϕ on X is a flow on X^*.*

Theorem 2.3. [2] *Let X be a completely regular space. There exists a unique compact Hausdorff space $\beta(X)$ such that*

1. *X is a dense subset of $\beta(X)$ and*
2. *Given any continuous function $f : X \to Y$ of X into a compact Hausdorff space Y, the function f extends uniquely to a continuous function $F : \beta(X) \to Y$.*

Here, $\beta(X)$ is called the Stone-Čech compactification of X.

Lemma 2.4. *Let X be a completely regular space. A continuous map $f : X \to X \subset \beta(X)$ extends uniquely to a continuous map $F : \beta(X) \to \beta(X)$. Let U and V be open subsets of $\beta(X)$. If $f(X \cap U) \subset V$, then $F(U) \subset \overline{V}$.*

Proof. Let $x \in U \subset \beta(X) = \overline{X}$. There exists a net (x_λ) in X such that $x_\lambda \to x$. Since F is continuous, $F(x_\lambda) \to F(x)$. Since $x_\lambda \to x$ and U is a neighborhood of x, we may assume that $x_\lambda \in U$ for all λ. Since $F(x_\lambda) = f(x_\lambda) \in f(X \cap U) \subset V$ for all λ, we have $F(x) \in \overline{V}$. Thus $F(U) \subset \overline{V}$. □

Theorem 2.5. *Let X be a completely regular space. Suppose that $\beta(X)$ has a locally finite basis. A flow ϕ on X extends uniquely to a flow Φ on $\beta(X)$.*

Proof. For each $t \in \mathbb{R}$, since $\phi^t : X \to X \subset \beta(X)$ is a continuous map, there exists a unique continuous map $\Phi^t : \beta(X) \to \beta(X)$ such that $\Phi^t(x) = \phi^t(x)$ for all $x \in X$.
Define $\Phi : \beta(X) \times \mathbb{R} \to \beta(X)$ by $\Phi(x,t) = \Phi^t(x)$ for all $(x,t) \in \beta(X) \times \mathbb{R}$. Since

$$\mathrm{Id}_{\beta(X)}(x) = x = \phi^0(x) \text{ for all } x \in X,$$

by the uniqueness of extension, we have $\Phi^0 = \mathrm{Id}_{\beta(X)}$. Let $s, t \in \mathbb{R}$. Since

$$\Phi^s(\Phi^t(x)) = \Phi^s(\phi^t(x)) = \phi^s(\phi^t(x)) = \phi^{s+t}(x) \text{ for all } x \in X,$$

by the uniqueness of extension, we have $\Phi^s \circ \Phi^t = \Phi^{s+t}$. Thus Φ is a flow. We will show that Φ is continuous. Let $(x,t) \in \beta(X) \times \mathbb{R}$. Given any neighborhood U_0 of $\Phi(x,t) = \Phi^t(x)$, there exists a neighborhood U of $\Phi^t(x)$ such that $\overline{U} \subset U_0$. Since Φ^t is continuous, there exists a neighborhood V_0 of x such that $\Phi^t(V_0) \subset U$. Since $\beta(X)$ is a compact

Hausdorff space, there exists a neighborhood V of x such that $\overline{V} \subset V_0$. For each $y \in X \cap V$, we have

$$\phi(y,t) = \phi^t(y) = \Phi^t(y) \in \Phi^t(V) \subset \Phi^t(V_0) \subset U.$$

Since $\phi(y,t) \in X \cap U$ and ϕ is continuous at (y,t), there exists a neighborhood W_y of y and a neighborhood I_y of t such that

$$\phi(z,s) \in X \cap U \text{ for all } z \in X \cap W_y \text{ and all } s \in I_y.$$

For each $y \in X \cap V$, we can choose a basic open set B_y such that

$$y \in B_y \subset \overline{B_y} \subset W_y.$$

Let $B = \bigcup_{y \in X \cap V} B_y$. Then $X \cap V \subset B$. We claim that $\overline{V} \subset \overline{B}$. Assume that $\overline{V} \not\subset \overline{B}$. There exists $p \in \overline{V} - \overline{B}$. Since $\beta(X) - \overline{B}$ is a neighborhood of p, $V \cap (\beta(X) - \overline{B}) \neq \emptyset$. Let $q \in V \cap (\beta(X) - \overline{B})$. Since $q \in \beta(X) = \overline{X}$ and $V \cap (\beta(X) - \overline{B})$ is a neighborhood of q, we have $X \cap V \cap (\beta(X) - \overline{B}) \neq \emptyset$. Since $X \cap V \subset B$, we have a contradiction. Thus $\overline{V} \subset \overline{B}$. Since $\{B_y \mid y \in V \cap X\}$ is locally finite, we have

$$\overline{B} = \overline{\bigcup_{y \in X \cap V} B_y} = \bigcup_{y \in X \cap V} \overline{B_y} \subset \bigcup_{y \in X \cap V} W_y.$$

Thus $\overline{V} \subset \overline{B} \subset \bigcup_{y \in X \cap V} W_y$. Since $\overline{V}$ is compact, there exists finitely many $y_1, \ldots, y_n \in X \cap V$ such that $\overline{V} \subset \bigcup_{k=1}^n W_{y_k}$. Let $(z,s) \in V \times \bigcap_{k=1}^n I_{y_k}$. Then $z \in W_{y_k}$ for some k. Since $s \in I_{y_k}$, we have $\phi^s(X \cap W_{y_k}) \subset X \cap U \subset U$. By the Lemma 2.4, we have

$$\Phi^s(W_{y_k}) \subset \overline{U} \subset U_0 \text{ and so } \Phi(z,s) = \Phi^s(z) \in \Phi^s(W_{y_k}) \subset U_0.$$

Thus Φ is continuous at (x,t). □

References

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